ON EXTREME POINTS IN SEPARABLE CONJUGATE SPACES

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ABSTRACT

It is proved that every bounded closed and convex subset of an arbitrary conjugate separable Banach space is the closed convex hull of its extreme points.

By the classical Krein Milman theorem, every convex bounded and weak-star closed subset of a conjugate Banach space is the weak-star closed convex hull of its extreme points. In general, the assumption of weak-star closedness cannot be replaced by norm closedness. For instance the unit ball of c_0 is a closed bounded subset of $m = l^*$ and has no extreme points. However in the separable case we have:

THEOREM 1. *Every bounded closed and convex subset of an arbitrary conjugate separable Banach space X is the closed convex hull of its extreme points.**

This theorem gives a useful criterion for a Banach space of being not isomorphically embeddable in any separable conjugate Banach space.

COROLLARY. *The space $L(0,1)$ is not isomorphic with any subspace of any separable conjugate Banach space* (cf. Gelfand [2], Pełczyński [6]).

This follows from the fact that the unit cell in $L(0,1)$ has no extreme points.

The assertion of Theorem 1 in the case $X = l$ has been recently obtained by Lindenstrauss [5]. The proof of Theorem 1 is a slight modification of Lindenstrauss's proof. The special properties of l are replaced by the properties of X expressed in terms of inclinations d_n (used by Kadec for constructing a homeomorphism between separable conjugate spaces).

According to [5, Lemma 1] the Theorem 1 is reduced to the following

PROPOSITION. *If K is a bounded closed convex subset of a separable conjugate Banach space, X then the set* $\operatorname{ext} K$ *of all extreme points in K is non empty.*

Proof. Assume that $\|\cdot\|$ is an admissible norm of X such that for the sequences in the unit sphere $\{x \in X;\ \|x\| = 1\}$ the weak-star convergence coincides with the norm convergence (such a norm exists, see Kadec [3] and Klee [4]). Let $X = Z^*$ and let (z_n) be a linearly independent and linearly dense sequence in Z.

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* Since every complex-conjugate space, regarded as a real space is a closed subspace of the suitable read conjugate space, it is enough to restrict the exttention to the real case.

Let $L_n = \{x \in X : x(z_i) = 0 \text{ for } i \leqq n\}$. Then (L_n) is a decreasing sequence of linear subspaces of X, and $\bigcap_{n=1}^{\infty} L_n = \{0\}$. For any x in X let

$$d_n(x) = \inf_{u \in L_n} \| x - u \| \qquad \text{for } n = 1, 2, \cdots$$

We shall also consider the space l of absolutely summable real-valued sequences with the usual norm. For any $\xi = (\xi_k) \in l$ let us set

$$d_n(\xi) = \sum_{k=1}^{n} |\xi_k| \qquad \text{for } n = 1, 2, \cdots$$

Let V be either X or l_1 Denote

$$B^V = \text{the unit cell in space } V,$$

$$T_n^V(\varepsilon) = \{x \in B^V : d_n(x) \geqq \| x \| - \varepsilon\}, \qquad \text{for } \varepsilon > 0$$

LEMMA 1. (Kadec [3], Klee [4]). *There is a homeomorphism* $h: X \xrightarrow[\text{onto}]{} l$ *such that* $d_n(x) = d_n(hx)$ *for* $n = 1, 2, \cdots$. *This homeomorphism has obviously the property*: $h(B^X) = B^l$ *and* $h(T_n^X(\varepsilon)) = T_n^l(\varepsilon)$.

Since the set $T_n^l(\varepsilon)$ is contained in the ε-neighbourhood of the compact set $B^l \cap \{\xi \in l : \xi_k = 0 \text{ for } k > n\}$, we get

LEMMA 2. *The set* $T_n^l(\varepsilon)$ *admits a finite* 2ε-*net.*

LEMMA 3. *If* (p_n) *is a sequence of positive integers* ,(F_n) *is a decreasing sequence of closed sets with* $F_k \subset T_{p_k}^V(1/k)$, $k = 1, 2, \cdots$ *and* V *is either* l *or* X, *then the set* $F = \bigcap_{k=1}^{\infty} F_k$ *is non empty and compact.*

Proof. 1° Let $V = l$. Let $x_n \in F_n$. By Lemma 2, (x_n) is totally bounded, hence it has a cluster point $x_0 = \bigcap_{n=1}^{\infty} F_n$, i.e. $F \neq \emptyset$. From Lemma 2 it also follows that F is totally bounded, and since F is closed, it must be compact.

2° In the case $V = X$ the assertion follows from Lemma 1 and from 1°.

LEMMA 4. *If* K *is a closed convex subset of* X *with* $\sup_{x \in K} \| x \| = M \leqq 1$, *then for every* $\varepsilon > 0$ *there exists a closed face* F_1 *in* K *and a positive integer* n *such that* $F_1 \subset T_n^X(\varepsilon)$.

Proof. Take an y in K such that $\| y \| \geqq M - \varepsilon/4$. Let n be such that $d_n(y) \geqq M - \varepsilon/2$. Let f be a linear functional such that $f(x) \leqq d_n(x)$ for all $x \in X$ and $f(y) = d_n(y)$ (f is a supporting functional of the "cylinder" $\{x : d_n(x) \leqq 1\}$ at the point $y/d_n(y)$. By the Bishop Phelps theorem [1], there exists a $g \in X^*$ with $\| g - f \| \leqq \varepsilon/4M$ such that the face $F_1 = \{x \in K : g(x) = \sup_{u \in K} g(u)\}$ is nonempty. For $x \in F_1$ we have

$$d_n(x) \geqq f(x) \geqq g(x) - \|f-g\| \, \|x\| \geqq g(y) - \|f-g\| \, \|x\|$$

$$\geqq f(y) - \|f-g\| \, (\|x\| + \|y\|) \geqq M - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \geqq \|x\| - \varepsilon.$$

Hence $F_1 \subset T_n^X(\varepsilon)$. Q.E.D.

Proof of the proposition. Without loss of generality we may assume that $K \subset B^X$. By Lemma 4, there is a sequence (F_n) of closed faces of K and a sequence of integers (p_n) such that F_{n+1} is a face of F_n and $F_n \subset T_{p_n}^X(1/n)$. Hence, by Lemma 3 $F = \bigcap_{n=1}^{\infty} F_n$ is a nonempty compact face of K. By the Krein Milman theorem, ext $F \neq \emptyset$ and hence ext $K \neq \emptyset$. This concludes the proof.

PROBLEM. Let X be a (separable) Banach space with the property that every bounded closed convex subset of X is the closed convex hull of its extreme points. Must X be isomorphic with a closed linear subspace of a (separable) conjugate space?

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